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The effect of slip on the motion of a sphere close to a wall and of two adjacent spheres

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SUMMARY
The motion of a sphere towards a plane or another sphere is opposed by the fluid between them with a force which is inversely proportional to the gap. In consequence, it is impossible for a constant force to produce contact in a finite time, unless the Stokes equations are modified. When the gap is of the same order as the mean free path of the air molecules, the Stokes theory for the motion of the air must be modified. The Maxwell slip flow approximation is used in this paper to show that, when the gap is small, the resisting force between the approaching surfaces becomes only logarithmically dependent on the gap, and contact can be achieved in a finite time. The difficulty in applying the Stokes theory to the problem of determining collision efficiencies for cloud droplets is thereby removed.

The calculated values of the resistance to approach are used to determine the motion of a sphere falling towards a plane. If the motion is compared with the corresponding motion when no allowance is made for slip flow, the sphere without slip would still be at a distance of 1.3 times the mean free path from the plane, when the sphere with slip has made contact.

Transverse motion must also be considered if the trajectory of a particle close to a collector is required. The forces and couples on the sphere in that situation have a logarithmic dependence on the gap without slip, but they tend to constant values when the effect of slip is included. Some calculations of collision efficiency of drops falling under gravity (Hocking and Jonas [1]) have been amended to include the effect of slip when the colliding drops are very close together, and show a significant increase in the collision efficiency.

1. Introduction

Fluid between two surfaces resists their approach. If the motion is slow enough for the Reynolds number to be small, the Stokes equations predict that the resistance is inversely proportional to the gap between the surfaces, when this gap is small compared to the radii of curvature of the surfaces. The application of a finite force causing the surfaces to move closer to each other cannot, therefore, produce contact in a finite time. The Stokes equations are, of course, not valid when the gap becomes of molecular dimensions, and the exponential approach predicted by the Stokes theory rapidly leads to such small separations of the surfaces, when, also, it becomes difficult to define contact precisely.

When the normal motion of the surfaces is combined with a transverse relative motion, there are circumstances in which the failure of the Stokes theory becomes an embarrassment. One such case is the deposition of small particles carried in a stream of air onto a collector placed in the flow. Numerous calculations have been made of the trajectories of such particles in a variety of circumstances, but the details of the motion when the particle is very close to the collector are often ignored. The calculation of the efficiency of the collection process effectively assumes that a particle which approaches the collector to a distance which is comparable to the particle radius will collide with it. When the size of the collector is many times the size of the particles, the theory is probably sufficiently accurate. But in the similar problem of coalescence of cloud droplets, it is certainly not sufficient to ignore the mutual interaction of the two coalescing particles as in the region of most interest to the question of the growth of cloud droplets by this process, the particles are of comparable size. The approach of the two particles is provided by their differential fall speed under gravity and the failure of the Stokes theory to predict contact in a finite time now causes difficulties. Under the theory, the particles will steadily approach each other at decreasing speeds measured along the line of centres, but gravity will cause the larger drop to fall round and past the smaller, and they will eventually separate. In calculating collision efficiencies, it is customary to make the arbitrary assumption that collisions will in fact occur whenever the gap between the surfaces becomes less than some arbitrarily chosen small length (Davis and Sartor, [2]; Hocking and Jonas, [1]). This procedure recognises that the Stokes theory cannot explain what is taking place when the gap becomes very small, but it is, to say the least, unsatisfactory. Some justification of the procedure is afforded by the small variation of the calculated collision efficiency with this arbitrarily chosen length when the collision efficiency is large. There is greater variation when the collision efficiency is small, but such small values do not affect the calculations of the growth rate of the larger drops, which is the main quantity of interest in the application to cloud physics. The scavenging of small particles by the cloud droplets, however, has a low efficiency, and to calculate the extent of this cleansing of the air in clouds, and the resultant deposition of the pollutant on the ground, a more accurate assessment of the arbitrarily chosen length is required.

A complete discussion of the final stages of the approach of two liquid drops would be extremely complicated. When the gap is not too small, it is reasonable to assume that the drops are effectively rigid spheres and that the Stokes theory holds. As the gap decreases, a large number of effects could become important, such as the deformation of the drops, internal circulation within them, the compressibility of the air, rarefied gas effects and inter-molecular and electrical forces. For drops with radius in the range 10 μm–100 μm, which is the significant range in the cloud physics application, the necessary modification to the Stokes theory when the gap becomes comparable with the mean free path of the air molecules is the first effect to become important. The mean free path is approximately 0.1 μm, and it was shown in Hocking and Jonas [1] that drop deformation, internal circulation and electrical effects of magnitude typical of naturally occurring clouds were all unimportant at this size gap. The results of section 2 show that compressibility of the air is only important if

$$G = \mu_0 Wa/p_0 h^2 = O(1) \text{ or more}, \tag{1.1}$$

where p_0 and μ_0 are the pressure and viscosity of the air, a is the drop radius, h is the gap and W is the velocity of approach of the colliding drops. For a drop falling towards a fixed plane, the value of W can be found by equating the weight of the drop to the drag on the drop, which for small values of h/a gives

$$\tfrac{4}{3}\pi\rho_s a^3 g = 6\pi\mu_0 a^2 W/h,$$

and hence

$$G = 2a^2\rho_s g/9p_0 h,$$

where ρ_s is the density of the drop and g the gravitational acceleration. Thus the gap must be reduced to 2×10^{-4} μm for 100 μm drops and to 2×10^{-6} μm for 10 μm drops before any notice need be taken of the compressibility of the air.

The inter-molecular London forces are effective in promoting collisions between waterborne 1 μm particles in shear flow (Curtis and Hocking, [3]), but if these forces are to be important in the present problem their magnitude must be comparable with the weight of the drops, that is,

$$H/16\pi a^2 h^2 \rho_s g = O(1) \text{ or more}, \tag{1.2}$$

where $H=5\times10^{-20}$ J is the Hamaker constant for water in air. Hence the inter-molecular forces become important when the gap is less than 3×10^{-3} μm for 100 μm drops and less than 3×10^{-2} μm for 10 μm drops. It follows that 10 μm is the lower end of the range of drop size for which the effects of allowing for the mean free path dominate the effect of inter-molecular forces.

The modification of the Stokes theory when the gap is comparable with the mean free path is obtained in this paper. The main result is that this effect by itself reduces the resistance to approach of two surfaces to an extent which permits contact in a finite time. The last stages of the motion as the gap tends to zero would require all the other effects mentioned above to be included to give a complete description. However, the details of this stage in the motion are not important, compared with the reduction of the time to contact from infinity to a finite value which permits the calculation of collision efficiencies without the need to impose any arbitrary minimum gap condition for collision.

The simplest way of estimating the change in the flow, when the lengths over which changes in velocity occur are comparable with the mean free path, is to use the Maxwell slip-flow approximation. The continuum equations are retained, but the boundary condition of no slip is replaced by the condition that the relative velocity at the boundary be proportional to the tangential stress there. The constant of proportionality is not exactly defined, but is of the same order as the mean free path. (For a discussion of the validity of Maxwell's slip flow, see Liu and Lees, [4]). At separations which are much smaller than the mean free path, the slip flow should be replaced by free molecule flow.

The effect of slip on the motion of a sphere towards a fixed plane is considered in section 2. The gap is supposed to be small compared with the radius of the sphere, so that lubrication theory can be used to determine the flow and the force on the sphere. The main result is that the force is no longer inversely proportional to the gap, but when the gap is smaller than the mean free path, it is inversely proportional to the mean free path, and only logarithmically dependent on the gap. It is, therefore, possible for collisions to occur in a finite time. The particular problem for a sphere falling under gravity towards a plane is examined and the time for it to fall from a short distance above the plane to contact is determined. If a comparison is made between the motion with and without slip, at the time when the sphere with slip has made contact with the plane, the sphere moving without slip would still be at height 1.3 times the mean free path above the plane. Corresponding results for the normal motion of two spheres are given in section 3.

The transverse motion of a sphere parallel to a plane suffers a resistance which depends on the logarithm of the gap. The forces and couples on a translating and rotating sphere have been calculated for slip flow. The forces and couples all tend to constant values when the gap is less than the mean free path. The values of the forces and couples are given in sections 4 and 5, and corresponding results for the motion of two spheres perpendicular to their line of centres in section 6.

The application of these results to the collisions of cloud droplets requires the recalculation of the trajectories, including the slip flow amendments to the sizes of the forces when the gap is small. It is difficult to estimate beforehand the size of the corresponding changes, as the inertia of the drops must be included in the calculations, but the simple problem of a sphere falling towards a plane suggested that the distance of the gap chosen to make collision certain was too small, and the resulting collision efficiencies in Hocking and Jonas [1] were underestimated. Some corrected calculations are described in section 7 which show a considerable increase in the collision efficiency.

2. Sphere moving normally towards a plane

We consider first the motion of a sphere normal to a fixed plane. In the neighbourhood of the point where the gap between the sphere and the plane is a minimum, lubrication theory can be used, since variations across the gap are much larger than variations parallel to the surfaces. This inner region must be matched to the outer region, but the leading term in the resistance to normal motion comes entirely from the inner region.

The use of lubrication theory and the matching of the inner region to the outer flow for a sphere moving normally and transversely in the vicinity of a plane or another sphere has been developed for flow without slip in a series of papers by O'Neill and Stewartson [5], Cooley and O'Neill [6], [7], O'Neill and Majumdar [8], Goldman, Cox and Brenner [9]. The methods of these authors have been adapted to include the effects of slip, and the papers will be referred to collectively by the symbol NS (for no slip).

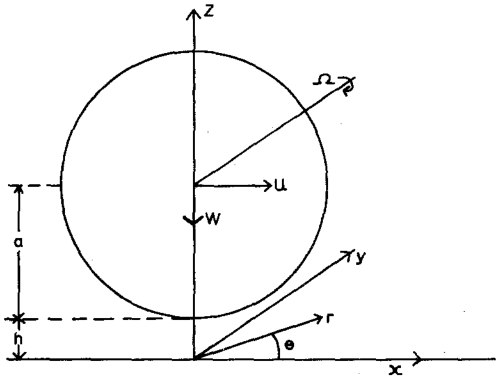

Figure 1. Definition sketch for sphere moving near a plane.

The quantities which specify the problem are the radius of the sphere a, the size of the gap h, the velocity of approach W, the mean free path of the air molecules λ, and the pressure, density, temperature and viscosity of the air at large distances from the sphere, which are respectively p_0, ρ_0, T_0 and μ_0. Cylindrical coordinates (r', z') are used, with origin on the fixed plane and with z' normal to the plane and passing through the centre of the sphere (see Fig. 1). The equation of the sphere surface is

$$z' = a+h-(a^2-r'^2)^{\frac{1}{2}}, \tag{2.1}$$

and the appropriate scaling of the space variables for the inner region is

$$z' = hz\,, \quad r' = (ah)^{\frac{1}{2}} r\,, \tag{2.2}$$

and the equation of the sphere surface then becomes

$$z = 1+\tfrac{1}{2}r^2+\tfrac{1}{8}\varepsilon r^4\,, \tag{2.3}$$

to first order in the small parameter $\varepsilon = h/a$. The radial and normal velocity components (u', w') are scaled by

$$u' = \varepsilon^{-\frac{1}{2}} Wu\,, \quad w' = Ww\,, \tag{2.4}$$

and the pressure, density, etc. are scaled by their values away from the sphere, p_0, ρ_0, etc. If the viscosity is assumed to be proportional to some power c of the absolute temperature, under this scaling we have

$$\mu = T^c\,, \tag{2.5}$$

with $\mu = T = 1$ away from the sphere. The Navier–Stokes equations and the energy equation with the variables scaled as described become

$$-\frac{\partial p}{\partial r} + G\frac{\partial}{\partial z}\left(T^c \frac{\partial u}{\partial z}\right) = G\,Re\left(\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial r} + \rho w\frac{\partial u}{\partial z}\right), \tag{2.6}$$

$$-\frac{\partial p}{\partial z} = 0\,, \tag{2.7}$$

$$\sigma\frac{\partial}{\partial z}\left(T^c\frac{\partial T}{\partial z}\right) + \frac{\gamma-1}{\gamma}G\,Re\left\{T^c\frac{\partial^2 u}{\partial z^2} + T^c\left(\frac{\partial u}{\partial z}\right)^2\right\} =$$

$$= Re\left(\rho u\frac{\partial T}{\partial r} + \rho w\frac{\partial T}{\partial z}\right) - \frac{\gamma-1}{\gamma}Re\left(u\frac{\partial p}{\partial r} + w\frac{\partial p}{\partial z}\right), \tag{2.8}$$

where σ^{-1} is the Prandtl number, γ the adiabatic index and

$$G = \frac{\mu_0 W a}{p_0 h^2}, \tag{2.9}$$

$$Re = \frac{\rho_0 W h}{\mu_0}. \tag{2.10}$$

Terms $O(\varepsilon)$ times those retained have been omitted.

The inertial terms on the right of (2.6) can be neglected if the Reynolds number $Re \ll 1$. Then the pressure variations, and hence the density variations, are $O(G)$ and since it was shown in section 1 that $G \ll 1$ for the drop sizes of main interest, the effect of compressibility can be ignored, giving

$$\rho = T = 1 \tag{2.11}$$

throughout the motion. The equation of continuity then has its incompressible form

$$\frac{\partial}{r\partial r}(ru) + \frac{\partial w}{\partial z} = 0. \tag{2.12}$$

The disappearance of all time derivatives when both G and Re are small means that the motion can be treated as quasi-steady, although the gap is continually changing. If G were $O(1)$ or more, the time dependence would have to be included in the equation of continuity and the problem would reduce to the solution of a partial, instead of ordinary, differential equation.

Since from (2.7), the pressure does not vary across the gap, p is a function of r only and (2.6) gives

$$Gu = \{\tfrac{1}{2}z^2 + zA(r) + B(r)\}(dp/dr). \tag{2.13}$$

The boundary conditions on the plane $z=0$ are

$$w = 0, \quad u - \tfrac{1}{6}\beta(\partial u/\partial z) = 0, \tag{2.14}$$

where $\beta = 6\lambda/h$. The slip velocity coefficient is not precisely defined, so that λ should be regarded as the mean free path multiplied by some numerical factor $O(1)$. On the sphere $z = 1 + \frac{1}{2}r^2$, the conditions are

$$w - ru = -1, \quad u + \tfrac{1}{6}\beta(\partial u/\partial z) = 0, \tag{2.15}$$

and when these conditions are applied to the value of u given by (2.13) and the corresponding value of w, determined from (2.12), we arrive at an equation for the pressure

$$\frac{d}{dx}\left[2x(1+x)^2(1+\beta+x)\frac{dp}{dx}\right] = -12G, \tag{2.16}$$

where $x = \frac{1}{2}r^2$. The conditions that p must satisfy are that p is finite at $x=0$ and $dp/dx \to 0$ as $x \to \infty$, and the appropriate solution of (2.16) is

$$p = p_1 - 6G\beta^{-2}\left[\log\frac{1+\beta+x}{1+x} - \frac{\beta}{1+x}\right], \tag{2.17}$$

where p_1 is the pressure at the inner boundary of the outer region.

The pressure is a maximum at $x=0$ and its value there varies between $p_1 + 3G$ when $\beta = 0$ and $p_1 + G/\beta$ when β is large. The validity of this solution depends on the Reynolds number being small and on the neglect of compressibility, that is

$$Re \ll 1, \quad G \ll \max(1, \beta). \tag{2.18}$$

The force resisting the approach of the sphere to the plane derives partly from the pressure and partly from the z-component of the tangential stress on the sphere. This second contribution however, can be shown to be $O(\varepsilon)$ compared to the first, which gives

$$f_N = \frac{p_0 h}{3\mu_0 W}\int_0^\infty (p - p_1)\,dx = \frac{a}{3hG}\int_0^\infty (p - p_1)\,dx, \tag{2.19}$$

where the z-component of the force on the sphere is $6\pi\mu_0 W a f_N$.

The contribution to the force from the outer region is of smaller order (see Colley and O'Neill, [7]) and need not be included. The value of the pressure given by (2.17) makes the force coefficient

$$f_N = \frac{2a}{h\beta^2}\{(1+\beta)\log(1+\beta)-\beta\}. \tag{2.20}$$

When $\beta \ll 1$, the force coefficient becomes a/h, in agreement with NS, but when $\beta \gg 1$ it is

$$f_N = \frac{a}{3\lambda}\log\frac{6\lambda}{eh}, \tag{2.21}$$

showing that the rapid rise of the force when the gap is reduced is slowed down by the slip to a logarithmic increase.

The value of f_N can be used to determine the time for contact to be achieved from a given initial separation, when the sphere is falling vertically under gravity towards a fixed horizontal plane. Because the gap is continually changing, so are the parameters G and β. If the inertia of the sphere is neglected, the velocity at any instant is determined by the balance between the resistance, which is proportional to the velocity, and the gravitational force. The equation of motion is

$$f_N \frac{dh}{dt} = -\frac{2(\rho_s-\rho_0)a^2 g}{9\mu_0}, \tag{2.22}$$

where ρ_s is the density of the sphere, and f_N is given by (2.20). With $h=6\lambda\eta$, (2.22) becomes

$$\left[(1+\eta)\log\left(1+\frac{1}{\eta}\right)-1\right]\frac{d\eta}{dt} = -\frac{(\rho_s-\rho_0)ag}{9\mu_0}, \tag{2.23}$$

so that the time τ for the gap to decrease from h_0 to 0 is

$$\tau = \frac{9\mu_0}{(\rho_s-\rho_0)ag}\int_0^{\eta_0}\left[(1+\eta)\log\left(1+\frac{1}{\eta}\right)-1\right]d\eta$$

$$= \frac{9\mu_0}{(\rho_s-\rho_0)ag}\left[\tfrac{1}{2}(1+\eta_0)^2\log\left(1+\frac{1}{\eta_0}\right)+\tfrac{1}{2}\log\eta_0-\tfrac{1}{2}\eta_0\right], \tag{2.24}$$

where $\eta_0 = h_0/6\lambda$. Without slip, $f_N = a/h$, and the time for the sphere to fall from a height h_0 to a height h_1 is

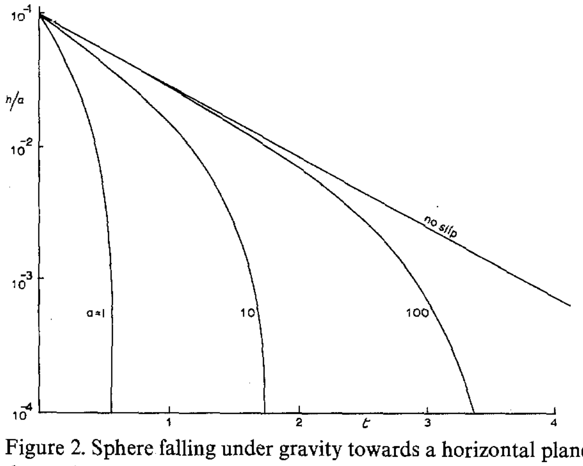

Figure 2. Sphere falling under gravity towards a horizontal plane, when the effect of slip is included. The straight line shows the corresponding heights when slip is not included. Mean free path is 0.1 μm. The time scale is 10^{-2} s when the radius $a=1$ μm, 10^{-3} s when $a=10$ μm and 10^{-4} s when $a=100$ μm.

$$\frac{9\mu_0}{(\rho_s-\rho_0)ag}\left[\tfrac{1}{2}\log\frac{h_0}{h_1}\right]. \tag{2.25}$$

When η_0 is large, the time for contact is approximately

$$\tau = \frac{9\mu_0}{(\rho_s-\rho_0)ag}\left[\tfrac{1}{2}\log\eta_0+\tfrac{3}{4}\right], \tag{2.26}$$

and with no slip, the sphere at this time would still be at a height 1.34λ above the plane.

The motion of spheres of three sizes are shown in Fig. 2, where the motion starts from a gap equal to one-tenth of the sphere radius. The mean free path was taken to be 0.1 μm. Also shown is the corresponding motion when slip is not included. It can be seen that the exponential approach of the sphere to the plane is speeded up as the gap becomes comparable with the mean free path.

3. Normal motion of two spheres

The effect of slip on the motion of two spheres along their line of centres can be found by a simple extension of the analysis for a sphere moving towards a plane. Suppose that there is a sphere of radius a_2 with centre at $z'=a_2+h$ and a sphere of radius a_1 with centre at $z'=-a_1$, and that the sphere of radius a_2 has a velocity W towards the second sphere, which is at rest. If the same scalings as in section 2 are made, namely (2.2) and (2.4), with a now defined by

$$a = \frac{a_1 a_2}{a_1+a_2}, \tag{3.1}$$

the equations of the two spheres in the lubrication region become

$$z=1+\tfrac{1}{2}\alpha r^2, \quad z=-\tfrac{1}{2}(1-\alpha)r^2, \tag{3.2}$$

where

$$\alpha = \frac{a_1}{a_1+a_2}. \tag{3.3}$$

If we let $a_1\to\infty$, we regain the special case of a sphere moving towards a plane, considered in section 2.

The boundary conditions on the two spheres are

$$w-\alpha r u = -1, \qquad u+\tfrac{1}{6}\beta\frac{\partial u}{\partial z}=0, \quad \text{on} \quad z=1+\tfrac{1}{2}\alpha r^2, \tag{3.4}$$

$$w+(1-\alpha)ru=0, \quad u-\tfrac{1}{6}\beta\frac{\partial u}{\partial z}=0, \quad \text{on} \quad z=-\tfrac{1}{2}(1-\alpha)r^2. \tag{3.5}$$

The analysis of section 2 can now be repeated and yields an equation for the pressure which is identical to (2.16). The solution (2.17) still holds and so does the expression (2.20) for the force on the sphere a_2. If the forces in the positive z-direction on the spheres of radius a_2 and a_1 are written as $6\pi\mu_0 a_2 W f_N^{(2)}$ and $6\pi\mu_0 a_1 W f_N^{(1)}$, respectively, the force coefficients are

$$f_N^{(2)} = \frac{2a_1^2 a_2}{(a_1+a_2)^2 h\beta^2}\left[(1+\beta)\log(1+\beta)-\beta\right], \tag{3.6}$$

$$f_N^{(1)} = -\frac{a_2}{a_1}f_N^{(2)}. \tag{3.7}$$

In the lubrication region, the forces depend only on the relative velocity of the two spheres, so that W is the velocity of approach when both spheres are in motion.

4. Sphere moving parallel to a plane

The calculation of the trajectory of a particle in the vicinity of a plane when it is moving towards and parallel to the plane requires the determination of the force resisting the transverse motion, and because there is usually no net couple on the sphere, the couple due to the translation as well as the force and couple on a rotating sphere must be calculated. With no slip, these forces and couples show a logarithmic dependence on the gap between the sphere and the plane. When the effect of slip is included, the calculations below show that the forces and couples tend to constant values when the gap is less than the mean free path. Because the inner region where the gap is narrowest is now less dominant over the rest of the flow when the forces are determined, it becomes necessary to match the inner region with the flow outside to obtain the forces and couples to terms $O(1)$ in the small parameter a/h. In the outer region the effect of slip can be ignored, since we are supposing that the radius of the sphere is much larger than the mean free path. The analysis of NS can be followed, the only changes necessary being the new forms of the boundary conditions in the inner region, allowing for slip.

If (r', θ, z') are cylindrical polar coordinates, with z' normal to the plane, the corresponding velocity components can be written $(u' \cos\theta, v' \sin\theta, w' \cos\theta)$, if the sphere has a velocity U in the direction $\theta = 0$ parallel to the plane and an angular velocity Ω about the axis $\theta = \pi/2$. The scaling of r' and z' is the same as used in section 2, namely (2.2), and the velocity scaling is

$$u' = Uu\,, \quad v' = Uv\,, \quad w' = \varepsilon^{\frac{1}{2}} Uw\,, \quad \Omega = U\omega/a\,, \tag{4.1}$$

where $\varepsilon = h/a$. If the pressure p' is written

$$p' = \mu_0 Ua\varepsilon^{\frac{1}{2}} h^{-2} p \cos\theta\,, \tag{4.2}$$

the equations of motion, with terms $O(\varepsilon)$ neglected, become

$$\frac{\partial p}{\partial z} = 0\,, \tag{4.3}$$

$$\frac{\partial p}{\partial r} = \frac{\partial^2 u}{\partial z^2}\,, \tag{4.4}$$

$$-\frac{p}{r} = \frac{\partial^2 v}{\partial z^2}\,, \tag{4.5}$$

$$\frac{\partial u}{\partial r} + \frac{u+v}{r} + \frac{\partial w}{\partial z} = 0\,, \tag{4.6}$$

and the boundary conditions are

$$u - \tfrac{1}{6}\beta \frac{\partial u}{\partial z} = v - \tfrac{1}{6}\beta \frac{\partial v}{\partial z} = w = 0 \quad \text{on} \quad z = 0\,, \tag{4.7}$$

$$\left.\begin{aligned} u + \tfrac{1}{6}\beta \frac{\partial u}{\partial z} = -\left(v + \tfrac{1}{6}\beta \frac{\partial v}{\partial z}\right) &= 1 - \omega \\ ru - w &= r \end{aligned}\right\} \quad \text{on} \quad z = 1 + \tfrac{1}{2}r^2\,. \tag{4.8}$$

With $p = rq$, these equations and boundary conditions lead to an equation for q which can be written

$$\begin{aligned} 2x(1+x)^2(1+\beta+x)\frac{d^2 q}{dx^2} + \{2x(1+x)(3+2\beta+3x) + 4(1+x)^2(1+\beta+x)\}\frac{dq}{dx} \\ + (1+x)(3+2\beta+3x)q = -6(1+\omega)\,, \end{aligned} \tag{4.9}$$

where $x = \frac{1}{2}r^2$, and the required solution must satisfy the conditions that q is finite at $x = 0$ and that $q = O(x^{-2})$ as $x \to \infty$. The latter condition is required to eliminate a solution which is too large at infinity to match with an acceptable solution in the outer region. When $\beta = 0$, the corresponding equation of NS has a very simple solution, which in the present notation is

$$q = \frac{6(1+\omega)}{5(1+x)^2}. \tag{4.10}$$

No such simple solution exists with $\beta \neq 0$, and the equation must be solved numerically. An asymptotic solution can be found when $\beta \gg 1$, and this solution is described below. The effect of compressibility can be neglected, provided

$$G' \equiv \frac{\mu_0 U a \varepsilon^{\frac{1}{2}}}{h^2 p_0} \ll \max(1, \beta). \tag{4.11}$$

The force on the sphere in the direction of the translation in the inner region is $-6\pi\mu_0 aUf^i$, where

$$f^i = \tfrac{1}{6}\int_0^r \left[rp + \frac{\partial u}{\partial z} - \frac{\partial v}{\partial z}\right]_{z=1+\frac{1}{2}r^2} r\,dr, \tag{4.12}$$

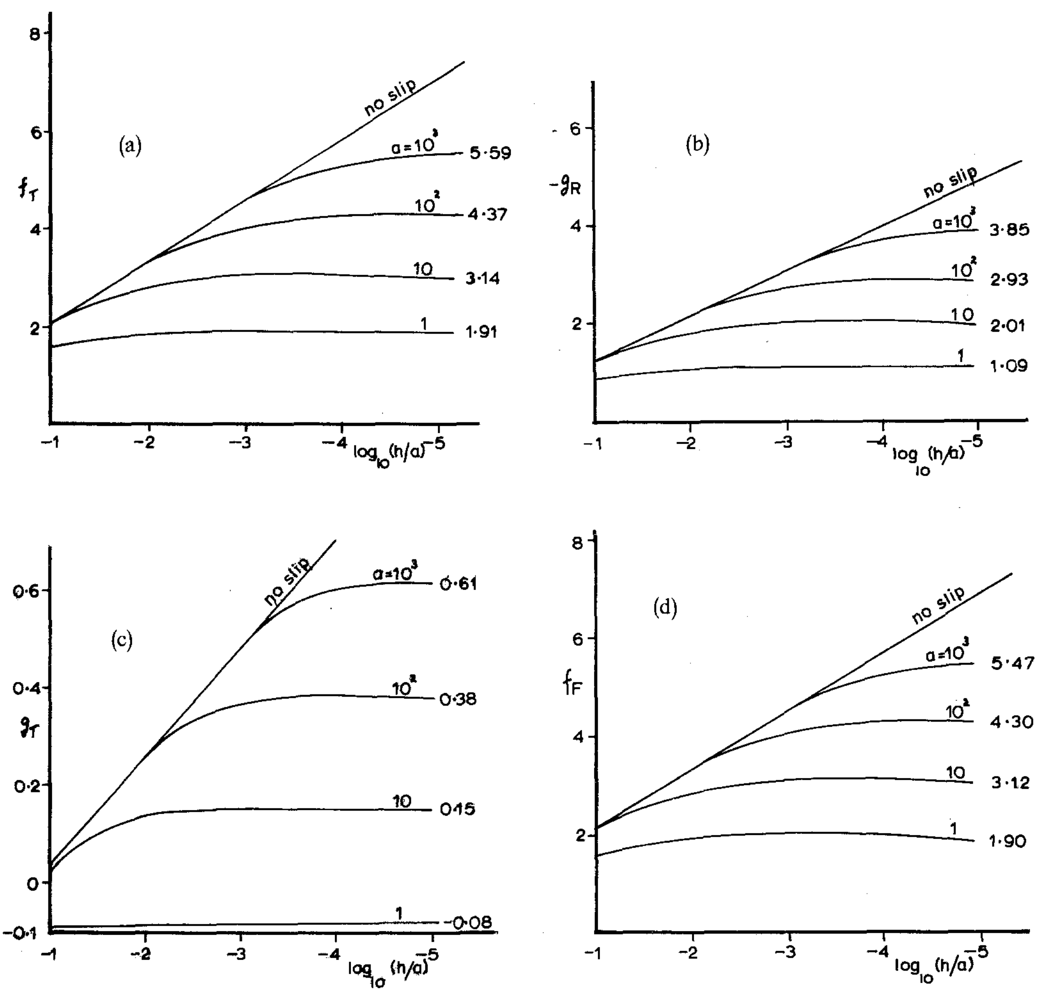

Figure 3. Force and couple coefficients for spheres of radius a μm moving parallel to a plane. The straight lines give the corresponding results with no slip. Also shown are the limiting values as $h/a \to 0$. Mean free path is 0.1 μm.
Figure 3(a). Force for translatory motion.
Figure 3(b). Couple for translatory motion. The force for rotatory motion is given by the relation $f_R = -4g_T/3$.
Figure 3(c). Couple for rotatory motion.
Figure 3(d). Force for motion under no couple.

(see O'Neill and Stewartson 1967) and substitution of the velocity components in terms of the pressure and integrating where possible gives

$$f^i = \tfrac{1}{3}(1-\omega)\log\frac{3+3x+\beta}{3+\beta} + \tfrac{1}{6}x^2 q(x) + \tfrac{1}{6}\int_0^x xq\,dx\,. \qquad (4.13)$$

The couple on the sphere can be written $8\pi\mu_0 a^2 U g^i$ and g^i determined similarly is

$$g^i = \tfrac{1}{4}(1-\omega)\log\frac{3+3x+\beta}{3+\beta} + \tfrac{1}{8}x^2 q(x) - \tfrac{1}{8}\int_0^x xq\,dx\,. \qquad (4.14)$$

These quantities must be evaluated for large x, where they are of the form $A \log x + B$, and added to the contributions to the force and couple for the outer region. These outer contributions are identical with those found in NS, and the combination eliminates the term in $\log x$. The terms independent of ω in (4.13) and (4.14) give the contributions to the force and couple from the translation and the parts with ω as a factor give the corresponding contributions from the rotation. The total expressions for the four coefficients are denoted by f_T, g_T, f_R and g_R, where the letters f and g refer to forces and couples, respectively, and the suffixes denote translation and rotation. The coefficients g_T and f_R are related by $3f_R + 4g_T = 0$ (Happel and Brenner, [10]). The calculated values of these coefficients are shown in Fig. 3(a, b, c) for various size spheres and for various gaps. If the sphere is free to rotate it will move so that the net couple on it is zero. The corresponding force on the sphere is denoted by f_F, where

$$f_F = f_T - f_R g_T / g_R\,, \qquad (4.15)$$

and the values of this coefficient are shown in Fig. 3(d). The no slip values of the coefficients are also shown on these figures and were derived from the results given by O'Neill and Stewartson [5] and Cooley and O'Neill [6].

5. Asymptotic solution for large β

An asymptotic solution of (4.9) for large values of β can be found. As pointed out earlier, at large values of β slip flow must give way to free molecule flow and the results of this section are included to complete the mathematical solution but they are not necessarily physically relevant. We consider separately the three regions $x \ll \beta$, $1 \ll x < \beta$, $x > \beta$. It is convenient to scale out the factor $6(1+\omega)$ from the right-hand side of the equation.

For the two outer regions, we write $x = \beta X$, $q = Q/\beta^2$, and, retaining the largest terms only, (4.9) becomes

$$X^2\left[2X^2\frac{d^2Q}{dX^2} + 10X\frac{dQ}{dX} + 3Q\right] + X\left[2X^2\frac{d^2Q}{dX^2} + 8X\frac{dQ}{dX} + 2Q\right] = -1\,, \qquad (5.1)$$

and we require the solution which satisfies

$$Q \sim \tfrac{1}{5}X^{-2} \text{ as } X \to \infty\,. \qquad (5.2)$$

The solution in descending powers of X is

$$\begin{aligned} Q = Q_1 + AQ_2 &= \tfrac{1}{5}X^{-2}\,{}_3F_2\left(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{5}, \tfrac{1}{2}-\tfrac{1}{2}\sqrt{5}, 1;\ 1+\sqrt{\tfrac{5}{2}}, 1-\sqrt{\tfrac{5}{2}};\ -X^{-1}\right) \\ &\quad + AX^{-2-\sqrt{(5/2)}}\,{}_2F_1\left(\tfrac{1}{2}+\sqrt{\tfrac{5}{2}}+\tfrac{1}{2}\sqrt{5}, \tfrac{1}{2}+\sqrt{\tfrac{5}{2}}-\tfrac{1}{2}\sqrt{5};\ 1+2\sqrt{\tfrac{5}{2}};\ -X^{-1}\right), \end{aligned} \qquad (5.3)$$

using the standard notation for generalised hypergeometric functions. Both parts of the solution are bounded for $X \geq 1$, but not for $X < 1$, so that the correct continuation into the middle region must be used. The omitted part of the complete solution of (5.1) has a leading term $X^{-2+\sqrt{(5/2)}}$ which violates the required behaviour for large X.

The analytic continuation of the ${}_2F_1$ function is a standard result and this part of Q can be written

$$Q_2 = Q_{21} + Q_{22} \qquad (5.4)$$

where

$$Q_{21} = \frac{\Gamma(1+2\sqrt{\frac{5}{2}})\,\Gamma(-\sqrt{5})}{\{\Gamma(\frac{1}{2}+\sqrt{\frac{5}{2}}-\frac{1}{2}\sqrt{5})\}^2}\, X^{(-3+\sqrt{5})/2}\, {}_2F_1(\tfrac{1}{2}+\sqrt{\tfrac{5}{2}}+\tfrac{1}{2}\sqrt{5},\ \tfrac{1}{2}-\sqrt{\tfrac{5}{2}}+\tfrac{1}{2}\sqrt{5};\ 1+\sqrt{5};\ -X) \tag{5.5}$$

and Q_{22} is a similar expression with the sign of $\sqrt{5}$ changed. The analytic continuation of Q_1 can be found in the usual way by writing down the Mellin–Barnes integral and changing the contour. The final result gives the value of Q in the range $X<1$ to be

$$\begin{aligned} Q = {} & \tfrac{1}{2}X^{-1}\, {}_3F_2(1+\sqrt{\tfrac{5}{2}};\ 1-\sqrt{\tfrac{5}{2}},\ 1;\ \tfrac{3}{2}+\tfrac{1}{2}\sqrt{5},\ \tfrac{3}{2}-\tfrac{1}{2}\sqrt{5};\ -X) \\ & + \left[A + \frac{\pi\sqrt{\frac{5}{2}}\,\Gamma(\frac{1}{2}+\sqrt{\frac{5}{2}}-\frac{1}{2}\sqrt{5})}{5\sin\pi\sqrt{\frac{5}{2}}\,\Gamma(\frac{1}{2}-\sqrt{\frac{5}{2}}-\frac{1}{2}\sqrt{5})\,\Gamma(1+2\sqrt{\frac{5}{2}})}\right] Q_{21} \\ & + \left[A + \frac{\pi\sqrt{\frac{5}{2}}\,\Gamma(\frac{1}{2}+\sqrt{\frac{5}{2}}+\frac{1}{2}\sqrt{5})}{5\sin\pi\sqrt{\frac{5}{2}}\,\Gamma(\frac{1}{2}-\sqrt{\frac{5}{2}}+\frac{1}{2}\sqrt{5})\,\Gamma(1+2\sqrt{\frac{5}{2}})}\right] Q_{22}\,. \end{aligned} \tag{5.6}$$

As $X \to 0$, the leading term is $O(X^{(-3-\sqrt{5})/2})$. If we revert to the original variables x and q, the solution in the range $x \ll \beta$ must satisfy the equation

$$2x(1+x)^2 \frac{d^2q}{dx^2} + [4x(1+x)+4(1+x)^2]\frac{dq}{dx} + 2(1+x)q = -1/\beta\,, \tag{5.7}$$

and the conditions

$$q \sim \beta^{(-1+\sqrt{5})/2} x^{(-3-\sqrt{5})/2} \quad \text{for} \quad x \gg 1\,, \tag{5.8}$$

and q finite at $x=0$. The solutions of (5.7) can be found as series in descending powers of $(1+x)$ and the solution with the required behaviour for large x is

$$q = (1+x)^{(-3-\sqrt{5})/2}\left[1 + \sum_1^{\infty} a_n (1+x)^{-n}\right], \tag{5.9}$$

where

$$a_n = \frac{\Gamma(n+\frac{3}{2}+\frac{1}{2}\sqrt{5})\,\Gamma(n+\frac{1}{2}+\frac{1}{2}\sqrt{5})\,\Gamma(1+\sqrt{5})}{\Gamma(n+1+\sqrt{5})\,\Gamma(n+1)\,\Gamma(\frac{3}{2}+\frac{1}{2}\sqrt{5})\,\Gamma(\frac{1}{2}+\frac{1}{2}\sqrt{5})} \tag{5.10}$$

and this solution diverges when $x=0$. The coefficient of Q_{22} in (5.6) must therefore be zero, i.e.

$$A = -\frac{\pi\,\Gamma(\frac{1}{2}+\sqrt{\frac{5}{2}}+\frac{1}{2}\sqrt{5})}{\sqrt{10}\sin\pi\sqrt{\frac{5}{2}}\,\Gamma(\frac{1}{2}-\sqrt{\frac{5}{2}}+\frac{1}{2}\sqrt{5})\,\Gamma(1+2\sqrt{\frac{5}{2}})}\,. \tag{5.11}$$

The contributions to the force and couple from the three regions can now be found. The inner region $x \ll \beta$ only makes a contribution $O(1/\beta)$ and can be ignored. The contributions from the regions $X<1$ and $X>1$ can be found by integrating the series expansions of the hypergeometric functions, term by term. When the contributions to the forces from these regions are added to the values obtained in NS for the outer region, the asymptotic values of the force and couple coefficients when β is large are found to be

$$f_T = \tfrac{8}{15}\log\left(\frac{a}{6\lambda}\right) + 1.615\,, \tag{5.12}$$

$$f_R = -\tfrac{2}{15}\log\left(\frac{a}{6\lambda}\right) + 0.186\,, \tag{5.13}$$

$$g_T = \tfrac{1}{10}\log\left(\frac{a}{6\lambda}\right) - 0.140\,, \tag{5.14}$$

$$g_R = -\tfrac{2}{5}\log\left(\frac{a}{6\lambda}\right) - 0.867\,, \tag{5.15}$$

and for a sphere falling under no net couple, the force coefficient is

$$f_F = \tfrac{1}{2}\log\left(\frac{a}{6\lambda}\right) + 1.77 - \frac{0.35}{\log\left(\frac{a}{6\lambda}\right) + 2.15}. \tag{5.16}$$

As pointed out by Goldman, Cox and Brenner [9], a sphere of radius a falling freely with speed U very close to a plane would have an angular velocity $U/4a$ on the incompressible theory. Allowing for slip, the angular velocity approaches a value depending on the ratio of the mean free path to the sphere radius and is always less than $U/4a$, and may be of opposite sign.

Experiments of Curty, quoted by Goldman, Cox and Brenner [9] on the rate of fall of spheres in the vicinity of planes indicated a surprising uniformity over a wide range of materials, fluids and sphere sizes, the low Reynolds number values all being given by

$$f_F = 8.96\,. \tag{5.17}$$

If the present results were applicable, much a high value of the resistance would imply a mean free path 10^{-7} times the sphere radius, which is far too small. It is noteworthy that the force calculated here is independent of the distance of the sphere from the plane, but the present theory is not appropriate to the experiments since the mean free path is much smaller than the likely size of irregularities on the surfaces, as pointed out by Goldman *et al.*

6. Transverse motion of two spheres

The extension of the analysis of section 4 to the motion of two spheres parallel to their line of centres can be achieved in a similar way to the corresponding extension for normal motion. The geometry is the same as in section 3, with a sphere of radius a_2 with its centre at $z=h+a_2$ and a sphere of radius a_1 with centre at $z=-a_1$. It is sufficient to suppose that the sphere a_1 is at rest and that the sphere a_2 has a velocity U in the x-direction and an angular velocity Ω in the y-direction. As in section 3, the length a is defined by

$$a = \frac{a_1 a_2}{a_1+a_2} \tag{6.1}$$

and the scalings (4.1) and (4.2) repeated, with the exception

$$\Omega = U\omega/a_2\,. \tag{6.2}$$

The boundary conditions become

$$u-\tfrac{1}{6}\beta\frac{\partial u}{\partial z} = v-\tfrac{1}{6}\beta\frac{\partial v}{\partial z} = w+(1-\alpha)ru = 0 \qquad \text{on} \quad z=-\tfrac{1}{2}(1-\alpha)r^2\,, \tag{6.3}$$

$$\left.\begin{aligned} u+\tfrac{1}{6}\beta\frac{\partial u}{\partial z} = -\left(v+\tfrac{1}{6}\beta\frac{\partial v}{\partial z}\right) &= 1-\omega \\ \alpha ru - w &= \alpha r \end{aligned}\right\} \quad \text{on} \quad z = 1+\tfrac{1}{2}\alpha r^2\,, \tag{6.4}$$

where, as before

$$\alpha = \frac{a_1}{a_1+a_2}\,. \tag{6.5}$$

The equations (4.3) to (4.6) still hold and their solution gives an equation for q which is identical with (4.9) except that the right-hand side is $-6(2\alpha-1+\omega)$ instead of $-6(1+\omega)$.

The contribution of the inner region to the forces and couples on the spheres can be found in a similar way to section 4. If the forces on the spheres in the x-direction are $-6\pi\mu_0 aUf_i^{(1)}$ and $-6\pi\mu_0 aUf_i^{(2)}$,

$$-f_i^{(1)} = f_i^{(2)} = \tfrac{1}{3}(1-\omega)\log\frac{3+3x+\beta}{3+\beta} + \tfrac{1}{6}x^2 q + \tfrac{1}{6}(2\alpha-1)\int_0^x xq\,dx\,. \tag{6.6}$$

Similarly, if the *y*-component of the couples on the spheres are $8\pi\mu_0 aa_2 Ug_i^{(2)}$ and $8\pi\mu_0 aa_1 Ug_i^{(1)}$,

$$g_i^{(1)} = \tfrac{1}{4}(1-\omega)\log\frac{3+3x+\beta}{3+\beta} - \tfrac{1}{8}x^2 q + \tfrac{1}{8}\int_0^x xq\,dx\,,$$

$$g_i^{(2)} = \tfrac{1}{4}(1-\omega)\log\frac{3+3x+\beta}{3+\beta} + \tfrac{1}{8}x^2 q - \tfrac{1}{8}\int_0^x xq\,dx\,. \qquad (6.7)$$

The contributions from the outer region must be added to these values to give the complete expressions for the forces and couples. These outer contributions are the same as when there is no slip, but they have not been fully derived. O'Neill and Majumdar [8] give the terms O (log (a/h)) but not the $O(1)$ terms in the asymptotic values of the forces and couples for small h/a.

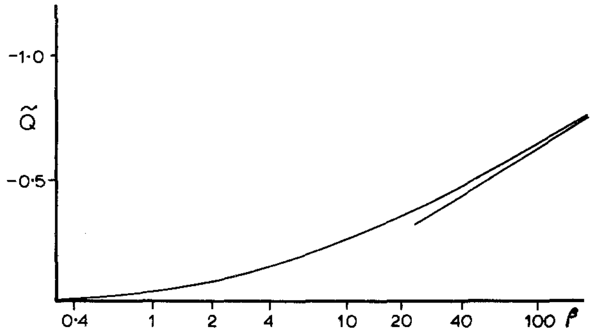

Figure 4. The function $\tilde{Q}(\beta)$.

However, the forces and couples at arbitrary values of h have been calculated numerically (O'Neill and Majumdar [8], Davis [11]) so it is sufficient to derive the *additional* contribution arising from the slip flow, which can be found by subtracting from (6.6) and (6.7) the values of these expressions when $\beta=0$. If $\tilde{Q}$ is defined by

$$\tilde{Q} = \int_0^\infty x\left[\tilde{q} - \frac{1}{5(1+x)^2}\right]dx\,, \qquad (6.8)$$

where $\tilde{q}$ is the solution of

$$2x(1+x)^2(1+\beta+x)\frac{d^2q}{dx^2} + [2x(1+x)(3+2\beta+3x)+4(1+x)^2(1+\beta+x)]\times$$

$$\times\frac{dq}{dx} + (1+x)(3+2\beta+3x)q = -1 \qquad (6.9)$$

which is finite at $x=0$ and which asymptotes to $1/(5x^2)$ as x tends to infinity, the value of $\tilde{Q}$ as a function of β can be determined numerically and is plotted in Figure 4. The forces on the spheres of radius a_1 and a_2 can be written

$$-6\pi\mu_0 aU(f_{NS}^{(1)} + f_S^{(1)})\,, \quad -6\pi\mu_0 aU(f_{NS}^{(2)} + f_S^{(2)})\,, \qquad (6.10)$$

respectively, where the suffix NS denotes the values when there is slip and the suffix S the additional contribution when slip is included. The couples can similarly be written

$$8\pi\mu_0 aa_1 U(g_{NS}^{(1)} + g_S^{(1)})\,, \quad 8\pi\mu_0 aa_2 U(g_{NS}^{(2)} + g_S^{(2)})\,, \qquad (6.11)$$

and the values of f_S and g_S are

$$-f_S^{(1)} = f_S^{(2)} = -\tfrac{1}{3}(1-\omega)\log(1+\tfrac{1}{3}\beta) + (2\alpha-1)(2\alpha-1+\omega)\tilde{Q}\,, \qquad (6.12)$$

$$g_S^{(1)} = -\tfrac{1}{4}(1-\omega)\log(1+\tfrac{1}{3}\beta) + \tfrac{3}{4}(2\alpha-1+\omega)\tilde{Q}\,, \qquad (6.13)$$

$$g_S^{(2)} = -\tfrac{1}{4}(1-\omega)\log(1+\tfrac{1}{3}\beta) - \tfrac{3}{4}(2\alpha-1+\omega)\tilde{Q}\,. \qquad (6.14)$$

The asymptotic analysis of O'Neill and Majumdar showed that f_{NS} and g_{NS} contain terms $O(\log(a/h))$ and the asymptotic analysis of section 5 showed that f_S and g_S contain terms $O(\log\beta)$ for large β. The logarithmic terms combine to give terms proportional to $\log(a/\lambda)$ so that, as h tends to zero, the force and couple coefficients tend to constant values.

7. The collision efficiency of small drops

In order to determine the changes in the spectrum of drop sizes in a cloud, it is necessary to know the collision efficiency for a pair of drops of different sizes. Collisions are produced by their different rates of fall under gravity and the efficiency of the collision process is defined as follows. Only the smaller drops which lie within a certain vertical cylinder below a larger drop will collide with it. If the two drops were to fall independently, the horizontal cross-section of this cylinder would have an area $\pi(a_1+a_2)^2$, where a_1 and a_2 are the two radii. The ratio of the actual cross-section within which collisions occur to the cross-section for independent motion is the collision efficiency, E. The calculation of E when both drops are sufficiently small for the Stokes equations to hold requires the computation of the relative trajectory of the smaller sphere, starting from an arbitrary horizontal separation and a large vertical separation and using the values of the forces and couples on two spheres moving arbitrarily. Details of the calculations can be found in Hocking and Jonas [1] but, because of the difficulty described in section 1, collisions were arbitrarily defined as occurring when the gap had become less than some small fraction ε of the drop radius. If the effects of allowing for slip flow when the gap becomes comparable with the mean free path are included, collisions can be precisely defined as occurring when the gap becomes zero. Since slip flow must be replaced by free molecule flow when the gap becomes very small, and since some of the other effects mentioned in section 1 must then also be included, the details of the final stage of the motion have not been accurately modelled, but such details are unlikely to affect the collision efficiency by a significant amount.

Some of the calculations of collision efficiencies made by Hocking and Jonas [1] have been repeated with the effect of slip included (Jonas [12], Davis [13]). With the radius ratio fixed at $\frac{1}{2}$, three different sizes for the larger drop were used. For a radius of 30 μm, the collision efficiency is increased from about 50 percent to 54 percent, and for a radius of 20 μm the increase is similar, but proportionally more significant. With $\varepsilon=10^{-4}$, the previous results gave $E=2.4$ percent and with $\varepsilon=10^{-3}$, $E=4.6$ percent, but with slip included, $E=7.3$ percent. The proportional increase is even more marked with a drop of radius 10 μm, for which E is increased from less than 1 percent to about 3.2 percent. The calculations for a 20 μm drop indicated that collision occurred when, without slip, the gap was 1.32 times the mean free path (compare section 2).

The main conclusions to be drawn from the results of this paper are that, for particles in relative motion in a fluid, it is important to include the effect of slip when they are sufficiently small (for water drops in air, less than about 20 μm radius) and that a rough guide to the effect of slip can be obtained by supposing that collisions will occur when, without slip, the gap becomes less than about 1.3 times the mean free path.

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